Math 582B
Homework Set 1
Due: March 6, 2007

1. Use the implicit function theorem to determine whether the equation $f(x, y) = 0$ can be solved for y as a function of x in a neighborhood of the indicated point (x_0, y_0) for each of the following.

   (a) $f(x, y) = x^2 - y^2, x_0 = y_0 = 0$
   (b) $f(x, y) = \sqrt{\ln(x + y)}, x_0 = 1.5, y_0 = -0.5$
   (c) $f(x, y) = \sin[\pi(x + y)] - 1, x_0 = y_0 = 1/4$
   (d) $f(x, y) = x^2 - y^2 - y, x_0 = y_0 = 0$

2. Suppose that

$$u(x, y) = \frac{x^4 + y^4}{x}$$
$$v(x, y) = \sin x + \cos y$$

   Near which points (x, y) can one solve for x, y in terms of u, v?

3. In the system

$$\begin{aligned} 3x + 2y + z^2 + u + v^2 &= 0 \\ 4x + 3y + z + u^2 + v + w + 2 &= 0 \\ x + z + 2 + u^2 + 2 &= 0 \end{aligned}$$

   discuss the solvability for u, v, w in terms of x, y, z near the $x = y = z = 0, u = v = 0, w = -2$.

4. Prove that the initial value problem

$$y' = yt + e^t, y(0) = 1$$

   has a solution.

5. Implement the function for Picard iteration in Mathematica that we discussed in class, and then use it to calculate the iterates $\phi_0, \phi_1, \ldots, \phi_5$ for the initial value problem in the previous exercise.

6. Use the Mathematica function `DSolve` to find the exact solution of the IVP in the previous exercise, and then plot the solution using Mathematica on the interval $[-1, 1]$.

7. Use Picard iteration to solve $y' = y, y(0) = 1$. Hints: (1) find the exact solution by direct integration first, so you have a way to check your answer; (2) Use Mathematica to generate the first several Picard iterations and then make a prediction of the form of the function $u(t)$ in

$$\phi_n = u(t) + O(t^n + 1)$$

   where $O(t^n)$ is a polynomial with terms of order t^n or higher, whose exact form you don't care about.