Introduction to Grades Eight Through Twelve

The standards for grades eight through twelve are organized differently from those for kindergarten through grade seven. In this section strands are not used for organizational purposes as they are in the elementary grades because the mathematics studied in grades eight through twelve falls naturally under discipline headings: algebra, geometry, and so forth. Many schools teach this material in traditional courses; others teach it in an integrated fashion. To allow local educational agencies and teachers flexibility in teaching the material, the standards for grades eight through twelve do not mandate that a particular discipline be initiated and completed in a single grade. The core content of these subjects must be covered; students are expected to achieve the standards however these subjects are sequenced.

Standards are provided for Algebra I, geometry, Algebra II, trigonometry, mathematical analysis, linear algebra, probability and statistics, advanced placement probability and statistics, and calculus. Many of the more advanced subjects are not taught in every middle school or high school. Moreover, schools and districts have different ways of combining the subject matter in these various disciplines. For example, many schools combine some trigonometry, mathematical analysis, and linear algebra to form a precalculus course. Some districts prefer offering trigonometry content with Algebra II.

Table 1, "Mathematics Disciplines, by Grade Level," reflects typical grade-level groupings of these disciplines in both integrated and traditional curricula. The lightly shaded region reflects the minimum requirement for mastery by all students. The dark shaded region depicts content that is typically considered elective but that should also be mastered by students who complete the other disciplines in the lower grade levels and continue the study of mathematics.

Many other combinations of these advanced subjects into courses are possible. What is described in this section are standards for the academic content by discipline; this document does not endorse a particular choice of structure for courses or a particular method of teaching the mathematical content.

When students delve deeply into mathematics, they gain not only conceptual understanding of mathematical principles but also knowledge of, and experience with, pure reasoning. One of the most important goals of mathematics is to teach students logical reasoning. The logical reasoning inherent in the study of mathematics allows for applications to a broad range of situations in which answers to practical problems can be found with accuracy.

By grade eight, students' mathematical sensitivity should be sharpened. Students need to start perceiving logical subtleties and appreciate the need for sound mathematical arguments before making conclusions. Students who are not prepared for Algebra I by grade nine should instead receive specialized instructional materials that focus on the prerequisite standards described in Appendix E. An algebra readiness course will prepare students for success in algebra and subsequent advanced courses. As students progress in the study of mathematics, they learn to distinguish between inductive and deductive reasoning; understand

Table 1. Mathematics Disciplines, by Grade Level

Disciplines	*Grades*				
	Eight	*Nine*	*Ten*	*Eleven*	*Twelve*
Algebra I					
Geometry					
Algebra II					
Probability and Statistics					
Trigonometry					
Linear Algebra					
Mathematical Analysis					
Advanced Placement Probability and Statistics					
Calculus					

the meaning of logical implication; test general assertions; realize that one counterexample is enough to show that a general assertion is false; understand conceptually that although a general assertion is true in a few cases, it may not be true in all cases; distinguish between something being proven and a mere plausibility argument; and identify logical errors in chains of reasoning.

Mathematical reasoning and conceptual understanding are not separate from content; they are intrinsic to the mathematical discipline students master at more advanced levels.

Algebra I Mathematics Content Standards

Symbolic reasoning and calculations with symbols are central in algebra. Through the study of algebra, a student develops an understanding of the symbolic language of mathematics and the sciences. In addition, algebraic skills and concepts are developed and used in a wide variety of problem-solving situations.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable:

1.1 Students use properties of numbers to demonstrate whether assertions are true or false.

2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

Simplify $\left(x^3 y^{\frac{1}{2}}\right)^6 \sqrt{xy}$.

3.0 Students solve equations and inequalities involving absolute values.

Solve for x: $3|x| + 5 > 7$.

For which values of x is $|x + 4| = |x| + 4$?

4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

For what values of x is the following inequality valid?

$5(x - 1) > 3x + 2$.

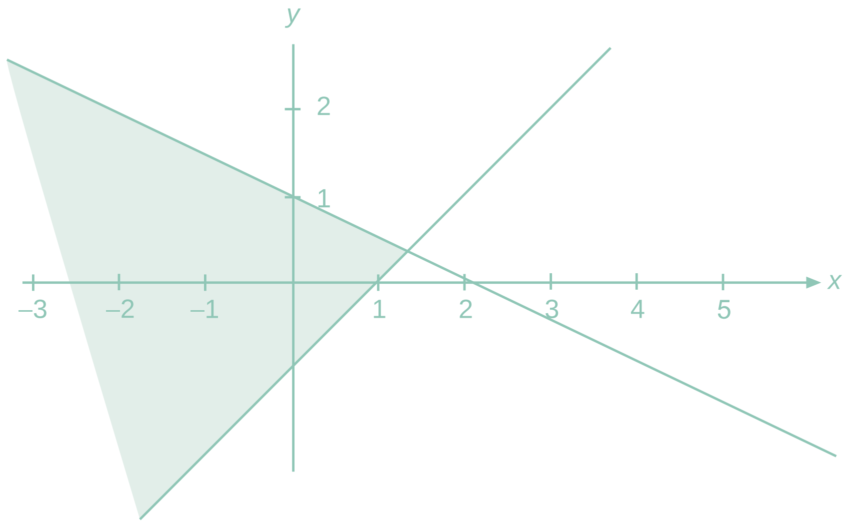
Expand and simplify $2(3x + 1) - 8x$.

5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

A-1 Pager Company charges a $25 set-up fee plus a $6.50 monthly charge. Cheaper Beeper charges $8 per month with no set-up fee. Set up an inequality to determine how long one would need to have the pager until the A-1 Pager plan would be the less expensive one.

6.0 Students graph a linear equation and compute the x- and y-intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by $2x + 6y < 4$).

Find inequalities whose simultaneous solution defines the region shown below:



7.0 Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.

Does the point (1, 2) lie on, above, or below the graph of the line $3x - 5y + 8 = 0$? Explain how you can be sure of your answer.

Write the equation of the line having x-intercept $-2\frac{1}{3}$ and y-intercept 5.

8.0 Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

Find the equation of the line passing through $(-1, \frac{1}{3})$ and parallel to the line defined by $5x + 2y = 17$. Also find the equation of the line passing through the same point but perpendicular to the line $5x + 2y = 17$.

9.0 Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

Solve and sketch the lines and the solution set:

$$3x + y = -1$$

$$x - \frac{1}{2}y = \frac{4}{3}$$

10.0 Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

11.0 Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

Factor $9x^3 + 6x^2 + x$.

12.0 Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.

Simplify $\frac{x^2+2x+1}{x^2-1}$.

13.0 Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

Solve for x and give a reason for each step: $\frac{2}{3x+1}+2=\frac{2}{3}$. (ICAS 1997, 6)

14.0 Students solve a quadratic equation by factoring or completing the square.

15.0 Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

The sum of the two digits of a number is 10. If 36 is added to it, the digits will be reversed. Find the number.

Two cars A and B move at constant velocity. Car A starts from P to Q, 150 miles apart, at the same time that car B starts from Q to P. They meet at the end of $1\frac{1}{2}$ hours. If car A moves 10 miles per hour faster than car B, what are their velocities?

16.0 Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.

17.0 Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

18.0 Students determine whether a relation defined by a graph, a set of ordered pairs, or a symbolic expression is a function and justify the conclusion.

19.0 Students know the quadratic formula and are familiar with its proof by completing the square.

Toni is solving this equation by completing the square.

$ax^2 + bx + c = 0$ (where $a \geq 0$)

Step 1. $ax^2 + bx = -c$

Step 2. $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Step 3. ?

Which response shown below should be step 3 in the solution?

1. $x^2 = -\frac{c}{b} - \frac{b}{a}x.$
2. $x + \frac{b}{a} = -\frac{c}{ax}.$
3. $x^2 + \frac{b}{a}x + \frac{b}{2a} = -\frac{c}{a} + \frac{b}{2a}.$
4. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$

(CST released test question, 2004)

20.0 Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

Suppose the graph of $y = px^2 + 5x + 2$ intersects the x-axis at two distinct points, where p is a constant. What are the possible values of p?

21.0 Students graph quadratic functions and know that their roots are the x-intercepts.

The graph of $y = x^2 + bx - 1$ passes through $(-\frac{1}{3}, 0)$

What is b?

22.0 Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x-axis in zero, one, or two points.

23.0 Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

24.0 Students use and know simple aspects of a logical argument:

24.1 Students explain the difference between inductive and deductive reasoning and identify and provide examples of each.

24.2 Students identify the hypothesis and conclusion in logical deduction.

24.3 Students use counterexamples to show that an assertion is false and recognize that a single counterexample is sufficient to refute an assertion.

25.0 Students use properties of the number system to judge the validity of results, to justify each step of a procedure, and to prove or disprove statements:

25.1 Students use properties of numbers to construct simple, valid arguments (direct and indirect) for, or formulate counterexamples to, claimed assertions.

25.2 Students judge the validity of an argument according to whether the properties of the real number system and the order of operations have been applied correctly at each step.

25.3 Given a specific algebraic statement involving linear, quadratic, or absolute value expressions or equations or inequalities, students determine whether the statement is true sometimes, always, or never.

Geometry Mathematics Content Standards

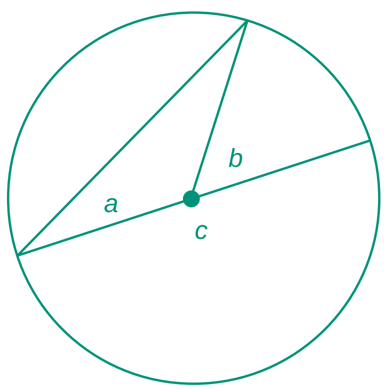
The geometry skills and concepts developed in this discipline are useful to all students. Aside from learning these skills and concepts, students will develop their ability to construct formal, logical arguments and proofs in geometric settings and problems.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning.

2.0 Students write geometric proofs, including proofs by contradiction.
If a line *L* is tangent to a circle at a point *P,* prove that the radius passing through *P* is perpendicular to *L*.

If *C* is the center of the circle in the figure shown below, prove that angle *b* has twice the measure of angle *a*.



3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

Prove or disprove: If two triangles have two pairs of congruent sides, the triangles must be congruent.

4.0 Students prove basic theorems involving congruence and similarity.

Prove that in a triangle, the larger angle faces the longer side.

If L_1, L_2, and L_3 are three parallel lines such that the distance from L_1 to L_2 is equal to the distance from L_2 to L_3, and if *l* is any transversal that intersects L_1, L_2, and L_3 at A_1, A_2, and A_3, respectively, prove that the segments A_1A_2 and A_2A_3 are congruent.

5.0 Students prove that triangles are congruent or similar, and they are able to use the concept of corresponding parts of congruent triangles.

Prove that a quadrilateral that has two pairs of congruent opposite angles is a parallelogram.

Prove that in $\triangle ABC$, if D is the midpoint of side AB and a line passing through D and parallel to BC intersects side AC at E, then E is the midpoint of side AC.

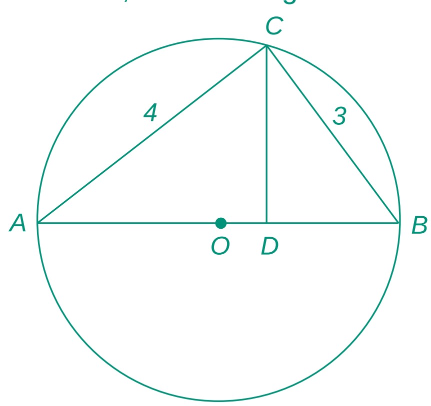
6.0 Students know and are able to use the triangle inequality theorem.

7.0 Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.

Prove that the figure formed by joining, in order, the midpoints of the sides of a quadrilateral is a parallelogram.

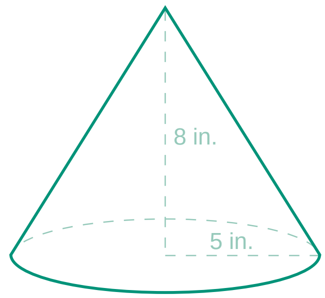
Using what you know about parallel lines cut by a transversal, show that the sum of the angles in a triangle is the same as the angle in a straight line, 180 degrees.

AB is a diameter of a circle centered at O. $CD \perp AB$. If the length of AB is 5, find the length of side CD.



8.0 Students know, derive, and solve problems involving the perimeter, circumference, area, volume, lateral area, and surface area of common geometric figures.

A right circular cone has radius 5 inches and height 8 inches.

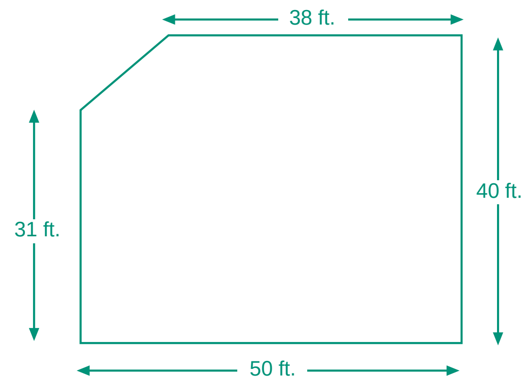


What is the lateral area of the cone? (Lateral area of cone = πrl, where l = slant height.) (CST released test question, 2004)

9.0 Students compute the volumes and surface areas of prisms, pyramids, cylinders, cones, and spheres; and students commit to memory the formulas for prisms, pyramids, and cylinders.

10.0 Students compute areas of polygons, including rectangles, scalene triangles, equilateral triangles, rhombi, parallelograms, and trapezoids.

The diagram below shows the overall floor plan for a house. It has right angles at three corners. What is the area of the house? What is the perimeter of the house? (CERT 1997, 26)



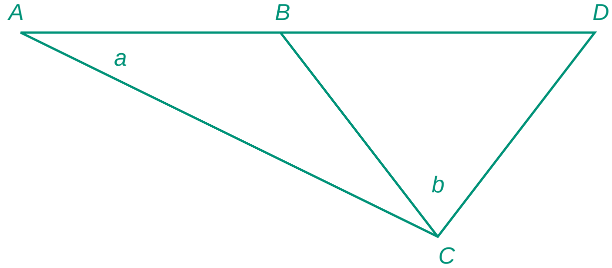
11.0 Students determine how changes in dimensions affect the perimeter, area, and volume of common geometric figures and solids.

A triangle has sides of lengths *a, b,* and *c* and an area *A.* What is the area of a triangle with sides of lengths 3*a*, 3*b*, and 3*c*, respectively? Prove that your answer is correct.

12.0 Students find and use measures of sides and of interior and exterior angles of triangles and polygons to classify figures and solve problems.

13.0 Students prove relationships between angles in polygons by using properties of complementary, supplementary, vertical, and exterior angles.

In the figure below, $\overline{AB} = \overline{BC} = \overline{CD}$. Find an expression for the measure of angle *b* in terms of the measure of angle *a* and prove that your expression is correct.



14.0 Students prove the Pythagorean theorem.

15.0 Students use the Pythagorean theorem to determine distance and find missing lengths of sides of right triangles.

16.0 Students perform basic constructions with a straightedge and compass, such as angle bisectors, perpendicular bisectors, and the line parallel to a given line through a point off the line.

Prove that the standard construction of the perpendicular from a point to a line not containing the point is correct.

17.0 Students prove theorems by using coordinate geometry, including the midpoint of a line segment, the distance formula, and various forms of equations of lines and circles.

Use coordinates to prove that if *ABC* is a triangle and *D, E* are points on sides *AB* and *AC,* respectively, so that

$$\frac{|AD|}{|AB|} = \frac{|AE|}{|AC|},$$

then line *DE* is parallel to *BC.*

18.0 Students know the definitions of the basic trigonometric functions defined by the angles of a right triangle. They also know and are able to use elementary relationships between them. For example, $\tan(x) = \sin(x)/\cos(x)$, $(\sin(x))^2 + (\cos(x))^2 = 1$.

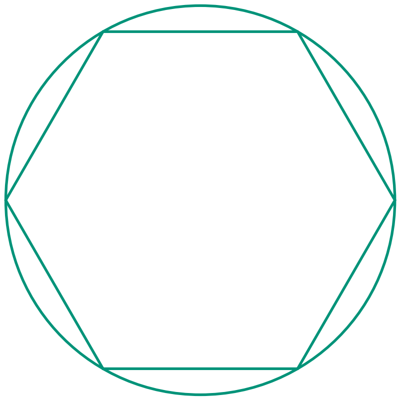
Without using a calculator, determine which is larger, tan(60°) or tan(70°) and explain why.

19.0 Students use trigonometric functions to solve for an unknown length of a side of a right triangle, given an angle and a length of a side.

20.0 Students know and are able to use angle and side relationships in problems with special right triangles, such as 30°, 60°, and 90° triangles and 45°, 45°, and 90° triangles.

21.0 Students prove and solve problems regarding relationships among chords, secants, tangents, inscribed angles, and inscribed and circumscribed polygons of circles.

Use the perimeter of a regular hexagon inscribed in a circle to explain why $\pi > 3$. (ICAS 1997,11)[4]



22.0 Students know the effect of rigid motions on figures in the coordinate plane and space, including rotations, translations, and reflections.

Use rigid motions to prove the side-angle-side criterion of triangle congruence.

[4] The Web site showing the source for the problems from the Intersegmental Committee of the Academic Senates (ICAS) is in the "Web Resources" section in "Works Cited."

Algebra II Mathematics Content Standards

This discipline complements and expands the mathematical content and concepts of Algebra I and geometry. Students who master Algebra II will gain experience with algebraic solutions of problems in various content areas, including the solution of systems of quadratic equations, logarithmic and exponential functions, the binomial theorem, and the complex number system.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students solve equations and inequalities involving absolute value.

Sketch the graph of each function.

$$y = \left|\frac{1}{x}\right|.$$

$$y = -\frac{2}{3}\left|x - 2\right| - 5.$$

2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or with matrices.

Draw the region in the plane that is the solution set for the inequality $(x - 1)(x + 2y) > 0$.

3.0 Students are adept at operations on polynomials, including long division.

Divide $x^4 - 3x^2 + 3x$ by $x^2 + 2$, and write the answer in the form:

$$\text{polynomial} + \frac{\text{linear polynomial}}{x^2 + 2}.$$

4.0 Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

Factor $x^3 + 8$.

5.0 Students demonstrate knowledge of how real and complex numbers are related both arithmetically and graphically. In particular, they can plot complex numbers as points in the plane.

6.0 Students add, subtract, multiply, and divide complex numbers.

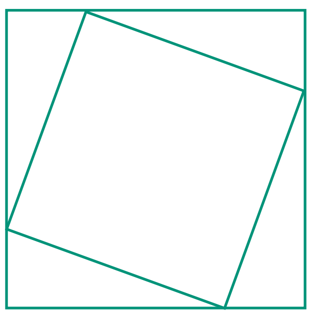
Write $\frac{1+i}{1-2i}$ in the form of $a + bi$, where a and b are real numbers.

7.0 Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

Simplify $\dfrac{\left(x^2 - x\right)^2}{x\left(x-1\right)^{-2}\left(x^2+3x-4\right)}$.

8.0 Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

In the figure shown below, the area between the two squares is 11 square inches. The sum of the perimeters of the two squares is 44 inches. Find the length of a side of the larger square. (ICAS 1997, 12)



9.0 Students demonstrate and explain the effect that changing a coefficient has on the graph of quadratic functions; that is, students can determine how the graph of a parabola changes as *a, b,* and *c* vary in the equation $y = a(x - b)^2 + c$.

10.0 Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

Find a quadratic function of x that has zeros at $x = -1$ and $x = 2$. Find a cubic equation of x that has zeros at $x = -1$ and $x = 2$ and nowhere else. (ICAS 1997, 7)

11.0 Students prove simple laws of logarithms.

11.1 Students understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Solve: $2^x = 5(13^{2x-5})$.

11.2 Students judge the validity of an argument according to whether the properties of real numbers, exponents, and logarithms have been applied correctly at each step.

12.0 Students know the laws of fractional exponents, understand exponential functions, and use these functions in problems involving exponential growth and decay.

Algebra II

The number of bacteria in a colony was growing exponentially. At 1 p.m. yesterday the number of bacteria was 100, and at 3 p.m. yesterday it was 4,000. How many bacteria were there in the colony at 6 p.m. yesterday? (TIMSS gr.12, K-13)

13.0 Students use the definition of logarithms to translate between logarithms in any base.

14.0 Students understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values.

1. Find the largest integer that is less than:

   $\log_{10}(1{,}256)$

   $\log_{10}(.029)$

2. $\frac{1}{2}\log_2 64 = ?$

15.0 Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

For positive numbers x and y, is the equation $\log_2 xy = \log_2 x \cdot \log_2 y$ always true, sometimes true, or never true?

If c is a real number, for what values of c is it true that $\frac{\sqrt{(c^2-1)^4}}{c+1} = c - 1$?

16.0 Students demonstrate and explain how the geometry of the graph of a conic section (e.g., asymptotes, foci, eccentricity) depends on the coefficients of the quadratic equation representing it.

What is the graph of $x^2 + py^2 - 4x + 10y - 26 = 0$ when $p = 1$? When $p = 4$? When $p = -4$?

17.0 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, students can use the method for completing the square to put the equation into standard form and can recognize whether the graph of the equation is a circle, ellipse, parabola, or hyperbola. Students can then graph the equation.

Does the origin lie inside, outside, or on the geometric figure whose equation is $x^2 + y^2 - 10x + 10y - 1 = 0$? Explain your reasoning. (ICAS 1997, 11)

18.0 Students use fundamental counting principles to compute combinations and permutations.

19.0 Students use combinations and permutations to compute probabilities.

20.0 Students know the binomial theorem and use it to expand binomial expressions that are raised to positive integer powers.

What is the third term of $(2x - 1)^6$? What is the general term? What is a simplified expression for the sum?

21.0 Students apply the method of mathematical induction to prove general statements about the positive integers.

Use mathematical induction to prove that for any integer $n \geq 1$, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

22.0 Students find the general term and the sums of arithmetic series and of both finite and infinite geometric series.

Find the sum of the arithmetic series: $13 + 16 + 19 + \ldots + 94$.

Find the sum of the geometric series:

$$\frac{3^5}{5^2} + \frac{3^6}{5^3} + \frac{3^7}{5^4} + \ldots + \frac{3^{32}}{5^{29}}.$$

23.0 Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series.

24.0 Students solve problems involving functional concepts, such as composition, defining the inverse function and performing arithmetic operations on functions.

Which of the following functions are their own inverse functions? Use at least two different methods to answer this question and explain your methods:

$$f(x) = \frac{2}{x} \qquad g(x) = x^3 + 4 \qquad h(x) = \frac{2 + \ln(x)}{2 - \ln(x)} \qquad j(x) = \sqrt[3]{\frac{x^3 + 1}{x^3 - 1}}$$

(ICAS 1997, 13)

25.0 Students use properties from number systems to justify steps in combining and simplifying functions.

Trigonometry Mathematics Content Standards

Trigonometry uses the techniques that students have previously learned from the study of algebra and geometry. The trigonometric functions studied are defined geometrically rather than in terms of algebraic equations. Facility with these functions as well as the ability to prove basic identities regarding them is especially important for students intending to study calculus, more advanced mathematics, physics and other sciences, and engineering in college.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students understand the notion of angle and how to measure it, in both degrees and radians. They can convert between degrees and radians.

2.0 Students know the definition of sine and cosine as *y*- and *x*-coordinates of points on the unit circle and are familiar with the graphs of the sine and cosine functions.

Find an angle β between 0 and 2π such that $\cos(\beta) = \cos(6\pi/7)$ and $\sin(\beta) = -\sin(6\pi/7)$. Find an angle θ between 0 and 2π such that $\sin(\theta) = \cos(6\pi/7)$ and $\cos(\theta) = \sin(6\pi/7)$.

3.0 Students know the identity $\cos^2(x) + \sin^2(x) = 1$:

3.1 Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity).

3.2 Students prove other trigonometric identities and simplify others by using the identity $\cos^2(x) + \sin^2(x) = 1$. For example, students use this identity to prove that $\sec^2(x) = \tan^2(x) + 1$.

Prove $\csc^2 x = 1 + \cot^2 x$.

4.0 Students graph functions of the form $f(t) = A \sin(Bt + C)$ or $f(t) = A \cos(Bt + C)$ and interpret *A*, *B*, and *C* in terms of amplitude, frequency, period, and phase shift.

On a graphing calculator, graph the function $f(x) = \sin(x) \cos(x)$. Select a window so that you can carefully examine the graph.

1. What is the apparent period of this function?
2. What is the apparent amplitude of this function?
3. Use this information to express *f* as a simpler trigonometric function.

5.0 Students know the definitions of the tangent and cotangent functions and can graph them.

6.0 Students know the definitions of the secant and cosecant functions and can graph them.

7.0 Students know that the tangent of the angle that a line makes with the *x*-axis is equal to the slope of the line.

8.0 Students know the definitions of the inverse trigonometric functions and can graph the functions.

9.0 Students compute, by hand, the values of the trigonometric functions and the inverse trigonometric functions at various standard points.

10.0 Students demonstrate an understanding of the addition formulas for sines and cosines and their proofs and can use those formulas to prove and/or simplify other trigonometric identities.

Use the addition formula for sine to find a numerical value of $\sin(75°)$.

Use the addition formula to find the numerical value of $\sin(15°)$.

Is $g(x) = 5 \sin 3x + 2 \cos x$ a periodic function? If so, what is its period? What is its amplitude?

11.0 Students demonstrate an understanding of half-angle and double-angle formulas for sines and cosines and can use those formulas to prove and/or simplify other trigonometric identities.

Express $\sin 3x$ in terms of $\sin x$ and $\cos x$.

12.0 Students use trigonometry to determine unknown sides or angles in right triangles.

13.0 Students know the law of sines and the law of cosines and apply those laws to solve problems.

A vertical pole sits between two points that are 60 feet apart. Guy wires to the top of that pole are staked at the two points. The guy wires are 40 feet and 35 feet long. How tall is the pole?

14.0 Students determine the area of a triangle, given one angle and the two adjacent sides.

Suppose in $\triangle ABC$ and $\triangle A'B'C'$, the sides of AB and $A'B'$ are congruent, as are AC and $A'C'$, but $\angle A$ is bigger than $\angle A'$. Which of $\triangle ABC$ and $\triangle A'B'C'$ has a bigger area? Prove that your answer is correct.

15.0 Students are familiar with polar coordinates. In particular, they can determine polar coordinates of a point given in rectangular coordinates and vice versa.

Trigonometry

16.0 Students represent equations given in rectangular coordinates in terms of polar coordinates.

Express the circle of radius 2 centered at (2, 0) in polar coordinates.

17.0 Students are familiar with complex numbers. They can represent a complex number in polar form and know how to multiply complex numbers in their polar form.

What is the angle that the ray from the origin to $3 + \sqrt{3}i$ makes with the positive x-axis?

18.0 Students know DeMoivre's theorem and can give nth roots of a complex number given in polar form.

19.0 Students are adept at using trigonometry in a variety of applications and word problems.

A lighthouse stands 100 feet above the surface of the ocean. From what distance can it be seen? (Assume that the radius of the earth is 3,960 miles.)

Mathematical Analysis Mathematics Content Standards

This discipline combines many of the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus and strengthens their conceptual understanding of problems and mathematical reasoning in solving problems. These standards take a functional point of view toward those topics. The most significant new concept is that of limits. Mathematical analysis is often combined with a course in trigonometry or perhaps with one in linear algebra to make a yearlong precalculus course.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students are familiar with, and can apply, polar coordinates and vectors in the plane. In particular, they can translate between polar and rectangular coordinates and can interpret polar coordinates and vectors graphically.

2.0 Students are adept at the arithmetic of complex numbers. They can use the trigonometric form of complex numbers and understand that a function of a complex variable can be viewed as a function of two real variables. They know the proof of DeMoivre's theorem.

3.0 Students can give proofs of various formulas by using the technique of mathematical induction.

Use mathematical induction to show that the sum of the interior angles in a convex polygon with n sides is $(n-2)\cdot 180°$.

4.0 Students know the statement of, and can apply, the fundamental theorem of algebra.

Find all cubic polynomials of x that have zeros at $x = -1$ and $x = 2$ and nowhere else. (ICAS 1997, 13)

5.0 Students are familiar with conic sections, both analytically and geometrically:

5.1 Students can take a quadratic equation in two variables; put it in standard form by completing the square and using rotations and translations, if necessary; determine what type of conic section the equation represents; and determine its geometric components (foci, asymptotes, and so forth).

5.2 Students can take a geometric description of a conic section—for example, the locus of points whose sum of its distances from (1, 0) and (–1, 0) is 6—and derive a quadratic equation representing it.

6.0 Students find the roots and poles of a rational function and can graph the function and locate its asymptotes.

7.0 Students demonstrate an understanding of functions and equations defined parametrically and can graph them.

8.0 Students are familiar with the notion of the limit of a sequence and the limit of a function as the independent variable approaches a number or infinity. They determine whether certain sequences converge or diverge.

Linear Algebra Mathematics Content Standards

The general goal in this discipline is for students to learn the techniques of matrix manipulation so that they can solve systems of linear equations in any number of variables. Linear algebra is most often combined with another subject, such as trigonometry, mathematical analysis, or precalculus.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students solve linear equations in any number of variables by using Gauss-Jordan elimination.

2.0 Students interpret linear systems as coefficient matrices and the Gauss-Jordan method as row operations on the coefficient matrix.

3.0 Students reduce rectangular matrices to row echelon form.

4.0 Students perform addition on matrices and vectors.

5.0 Students perform matrix multiplication and multiply vectors by matrices and by scalars.

6.0 Students demonstrate an understanding that linear systems are inconsistent (have no solutions), have exactly one solution, or have infinitely many solutions.

7.0 Students demonstrate an understanding of the geometric interpretation of vectors and vector addition (by means of parallelograms) in the plane and in three-dimensional space.

8.0 Students interpret geometrically the solution sets of systems of equations. For example, the solution set of a single linear equation in two variables is interpreted as a line in the plane, and the solution set of a two-by-two system is interpreted as the intersection of a pair of lines in the plane.

9.0 Students demonstrate an understanding of the notion of the inverse to a square matrix and apply that concept to solve systems of linear equations.

10.0 Students compute the determinants of 2×2 and 3×3 matrices and are familiar with their geometric interpretations as the area and volume of the parallelepipeds spanned by the images under the matrices of the standard basis vectors in two-dimensional and three-dimensional spaces.

11.0 Students know that a square matrix is invertible if, and only if, its determinant is nonzero. They can compute the inverse to 2×2 and 3×3 matrices using row reduction methods or Cramer's rule.

12.0 Students compute the scalar (dot) product of two vectors in n-dimensional space and know that perpendicular vectors have zero dot product.

Probability and Statistics Mathematics Content Standards

This discipline is an introduction to the study of probability, interpretation of data, and fundamental statistical problem solving. Mastery of this academic content will provide students with a solid foundation in probability and facility in processing statistical information.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students know the definition of the notion of *independent events* and can use the rules for addition, multiplication, and complementation to solve for probabilities of particular events in finite sample spaces.

2.0 Students know the definition of *conditional probability* and use it to solve for probabilities in finite sample spaces.

A whole number between 1 and 30 is chosen at random. If the digits of the number that is chosen add up to 8, what is the probability that the number is greater than 12?

3.0 Students demonstrate an understanding of the notion of *discrete random variables* by using them to solve for the probabilities of outcomes, such as the probability of the occurrence of five heads in 14 coin tosses.

4.0 Students are familiar with the standard distributions (normal, binomial, and exponential) and can use them to solve for events in problems in which the distribution belongs to those families.

5.0 Students determine the mean and the standard deviation of a normally distributed random variable.

6.0 Students know the definitions of the *mean, median,* and *mode* of a distribution of data and can compute each in particular situations.

7.0 Students compute the variance and the standard deviation of a distribution of data.

Find the mean and standard deviation of the following seven numbers:

4 12 5 6 8 5 9

Make up another list of seven numbers with the same mean and a smaller standard deviation. Make up another list of seven numbers with the same mean and a larger standard deviation. (ICAS 1997, 11)

8.0 Students organize and describe distributions of data by using a number of different methods, including frequency tables, histograms, standard line and bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.

Advanced Placement Probability and Statistics — Mathematics Content Standards

This discipline is a technical and in-depth extension of probability and statistics. In particular, mastery of academic content for advanced placement gives students the background to succeed in the *Advanced Placement* examination in the subject.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students solve probability problems with finite sample spaces by using the rules for addition, multiplication, and complementation for probability distributions and understand the simplifications that arise with independent events.

2.0 Students know the definition of *conditional probability* and use it to solve for probabilities in finite sample spaces.

You have 5 coins in your pocket: 1 penny, 2 nickels, 1 dime, and 1 quarter. If you pull out 2 coins at random and they are collectively worth more than 10 cents, what is the probability that you pulled out a quarter?

3.0 Students demonstrate an understanding of the notion of *discrete random variables* by using this concept to solve for the probabilities of outcomes, such as the probability of the occurrence of five or fewer heads in 14 coin tosses.

4.0 Students understand the notion of a *continuous random variable* and can interpret the probability of an outcome as the area of a region under the graph of the probability density function associated with the random variable.

Consider a continuous random variable X whose possible values are numbers between 0 and 2 and whose probability density function is given by $f(x) = 1 - \frac{1}{2}x$ for $0 \leq x \leq 2$. What is the probability that $X > 1$?

5.0 Students know the definition of the *mean of a discrete random variable* and can determine the mean for a particular discrete random variable.

6.0 Students know the definition of the *variance of a discrete random variable* and can determine the variance for a particular discrete random variable.

7.0 Students demonstrate an understanding of the standard distributions (normal, binomial, and exponential) and can use the distributions to solve for events in problems in which the distribution belongs to those families.

Suppose that X is a normally distributed random variable with mean $m = 0$. If $P(X < c) = \frac{2}{3}$, find $P(-c < X < c)$.

8.0	Students determine the mean and the standard deviation of a normally distributed random variable.
9.0	Students know the central limit theorem and can use it to obtain approximations for probabilities in problems of finite sample spaces in which the probabilities are distributed binomially.

Advanced Placement Probability and Statistics

10.0	Students know the definitions of the *mean, median,* and *mode of distribution* of data and can compute each of them in particular situations.
11.0	Students compute the variance and the standard deviation of a distribution of data.
12.0	Students find the line of best fit to a given distribution of data by using least squares regression.
13.0	Students know what the *correlation coefficient of two variables* means and are familiar with the coefficient's properties.
14.0	Students organize and describe distributions of data by using a number of different methods, including frequency tables, histograms, standard line graphs and bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.
15.0	Students are familiar with the notions of a statistic of a distribution of values, of the sampling distribution of a statistic, and of the variability of a statistic.
16.0	Students know basic facts concerning the relation between the mean and the standard deviation of a sampling distribution and the mean and the standard deviation of the population distribution.
17.0	Students determine confidence intervals for a simple random sample from a normal distribution of data and determine the sample size required for a desired margin of error.
18.0	Students determine the *P*-value for a statistic for a simple random sample from a normal distribution.
19.0	Students are familiar with the *chi*-square distribution and *chi*-square test and understand their uses.

Calculus Mathematics Content Standards

When taught in high school, calculus should be presented with the same level of depth and rigor as are entry-level college and university calculus courses. These standards outline a complete college curriculum in one-variable calculus. Many high school programs may have insufficient time to cover all of the following content in a typical academic year. For example, some districts may treat differential equations lightly and spend substantial time on infinite sequences and series. Others may do the opposite. Consideration of the College Board syllabi for the Calculus AB and Calculus BC sections of the *Advanced Placement Examinations in Mathematics* may be helpful in making curricular decisions. Calculus is a widely applied area of mathematics and involves a beautiful intrinsic theory. Students mastering this content will be exposed to both aspects of the subject.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students demonstrate knowledge of both the formal definition and the graphical interpretation of limit of values of functions. This knowledge includes one-sided limits, infinite limits, and limits at infinity. Students know the definition of convergence and divergence of a function as the domain variable approaches either a number or infinity:

1.1 Students prove and use theorems evaluating the limits of sums, products, quotients, and composition of functions.

1.2 Students use graphical calculators to verify and estimate limits.

1.3 Students prove and use special limits, such as the limits of $(\sin(x))/x$ and $(1 - \cos(x))/x$ as x tends to 0.

Evaluate the following limits, justifying each step:

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$$

$$\lim_{x\to 0} \frac{1-\cos(2x)}{\sin(3x)}$$

$$\lim_{x\to\infty} \left(x-\sqrt{x^2-x}\right)$$

2.0 Students demonstrate knowledge of both the formal definition and the graphical interpretation of continuity of a function.

For what values of x is the function $f(x) = \dfrac{x^2-1}{x^2-4x+3}$ continuous? Explain.

3.0 Students demonstrate an understanding and the application of the intermediate value theorem and the extreme value theorem.

4.0 Students demonstrate an understanding of the formal definition of the derivative of a function at a point and the notion of differentiability:

4.1 Students demonstrate an understanding of the derivative of a function as the slope of the tangent line to the graph of the function.

4.2 Students demonstrate an understanding of the interpretation of the derivative as an instantaneous rate of change. Students can use derivatives to solve a variety of problems from physics, chemistry, economics, and so forth that involve the rate of change of a function.

4.3 Students understand the relation between differentiability and continuity.

4.4 Students derive derivative formulas and use them to find the derivatives of algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic functions.

Find all points on the graph of $f(x) = \frac{x^2 - 2}{x + 1}$ where the tangent line is parallel to the tangent line at $x = 1$.

5.0 Students know the chain rule and its proof and applications to the calculation of the derivative of a variety of composite functions.

6.0 Students find the derivatives of parametrically defined functions and use implicit differentiation in a wide variety of problems in physics, chemistry, economics, and so forth.

For the curve given by the equation $\sqrt{x} + \sqrt{y} = 4$, use implicit differentiation to find $\frac{d^2y}{dx^2}$.

7.0 Students compute derivatives of higher orders.

8.0 Students know and can apply Rolle's theorem, the mean value theorem, and L'Hôpital's rule.

9.0 Students use differentiation to sketch, by hand, graphs of functions. They can identify maxima, minima, inflection points, and intervals in which the function is increasing and decreasing.

10.0 Students know Newton's method for approximating the zeros of a function.

11.0 Students use differentiation to solve optimization (maximum-minimum problems) in a variety of pure and applied contexts.

A man in a boat is 24 miles from a straight shore and wishes to reach a point 20 miles down shore. He can travel 5 miles per hour in the boat and 13 miles per hour on land. Find the minimal time for him to reach his destination and where along the shore he should land the boat to arrive as fast as possible.

12.0 Students use differentiation to solve related rate problems in a variety of pure and applied contexts.

13.0 Students know the definition of the definite integral by using Riemann sums. They use this definition to approximate integrals.

The following is a Riemann sum that approximates the area under the graph of a function $f(x)$, between $x = a$ and $x = b$. Determine a possible formula for the function $f(x)$ and for the values of a and b: $\sum_{i=1}^{n} \frac{2}{n} e^{1+\frac{2i}{n}}$.

14.0 Students apply the definition of the integral to model problems in physics, economics, and so forth, obtaining results in terms of integrals.

15.0 Students demonstrate knowledge and proof of the fundamental theorem of calculus and use it to interpret integrals as antiderivatives.

If $f(x) = \int_1^x \sqrt{1+t^3}\,dt$, find $f'(2)$.

16.0 Students use definite integrals in problems involving area, velocity, acceleration, volume of a solid, area of a surface of revolution, length of a curve, and work.

17.0 Students compute, by hand, the integrals of a wide variety of functions by using techniques of integration, such as substitution, integration by parts, and trigonometric substitution. They can also combine these techniques when appropriate.

Evaluate the following:

$$\int \frac{\sin\left(1-\sqrt{x}\right)}{\sqrt{x}}\,dx \qquad \int_1^e \frac{\ln x}{\sqrt{x}}\,dx \qquad \int_0^1 \sqrt{1+\sqrt{x}}\,dx$$

$$\int \arctan x\,dx \qquad \int \frac{\sqrt{x^2-1}}{x^3}\,dx \qquad \int \frac{dx}{e^x\sqrt{1-e^{2x}}}$$

Calculus

18.0 Students know the definitions and properties of inverse trigonometric functions and the expression of these functions as indefinite integrals.

19.0 Students compute, by hand, the integrals of rational functions by combining the techniques in standard 17.0 with the algebraic techniques of partial fractions and completing the square.

20.0 Students compute the integrals of trigonometric functions by using the techniques noted above.

21.0 Students understand the algorithms involved in Simpson's rule and Newton's method. They use calculators or computers or both to approximate integrals numerically.

22.0 Students understand improper integrals as limits of definite integrals.

23.0 Students demonstrate an understanding of the definitions of convergence and divergence of sequences and series of real numbers. By using such tests as the comparison test, ratio test, and alternate series test, they can determine whether a series converges.

Determine whether the following alternating series converge absolutely, converge conditionally, or diverge:

$$\sum_{n=3}^{\infty}(-1)^n\left(\frac{2^n}{n!}\right) \qquad \sum_{n=3}^{\infty}\frac{(-1)^n}{n\ln n} \qquad \sum_{n=3}^{\infty}(-1)^n\left(\frac{1+n}{n+\ln n}\right)$$

24.0 Students understand and can compute the radius (interval) of the convergence of power series.

25.0 Students differentiate and integrate the terms of a power series in order to form new series from known ones.

26.0 Students calculate Taylor polynomials and Taylor series of basic functions, including the remainder term.

27.0 Students know the techniques of solution of selected elementary differential equations and their applications to a wide variety of situations, including growth-and-decay problems.